UNIVERSALITY, SCALING AND TRIVIALITY IN A HIERARCHICAL SCALAR FIELD THEORY

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Using polynomial truncations of the Fourier transform of the RG transformation of Dyson's hierarchical model, we show that it is possible to calculate very accurately the renormalized quantities in the symmetric phase. Numerical results regarding the corrections to the scaling laws, triviality, hyperscaling, universality and high-accuracy determinations of the critical exponents are discussed.

1. Introduction

Probing shorter distance physics by direct production experiments is becoming increasingly difficult. In the 21-th century, we may have to look for small effects at accessible energies in order to learn about physics at higher energy. If this scenario is correct, it is important to develop methods of calculations which can outperform the accuracy of the Monte Carlo methods.

We present here some results showing that very simple *algebraic* methods can be used to calculate very accurately the renormalized quantities of Dyson's hierarchical model [1]. The RG transformation of this model is closely related [2] to Wilson's approximate recursion formula [3]. In both cases (which are examples of "hierarchical approximations"), the RG transformation maps a local potential into another local potential and there is no wave function renormalization or generation of higher derivative terms. Such a simplified version of the RG is only a *qualitative* approximation of the one which holds for nearest neighbor interaction lattice models. However, if we consider the values of the critical exponents, it is a much better approximation than the gaussian model.

As far as the zero momentum Green's functions are concerned, the numerical treatment that we proposed in Ref. [4] is tantamount to a closed form solution. The method advocated replaces the evaluation of multiple integrals (which in many lattice models can only be performed with the MC methods) by the evaluation of a *single* integral followed by purely algebraic manipulations. The computer time involved in this procedure grows only like the logarithm of the number of sites. These great numerical advantages lead us to reconsider the question of the improvment of the hierarchical approximation. This is a hard bookkeeping and computational problem which we plan to attack in the future.

Dyson's model couples the main spin in boxes of size 2^l with a strength $(\frac{c}{4})^l$, where c is a free parameter set to $2^{(D-2)/D}$ in order to approximate a nearest neighbor scalar model in D-dimensions. A more systematic presentation of the model can be found for instance in Ref. [4]. Models with fermi fields can be constructed similarly by replacing $D-2$ by $D-1$ in c. In addition one needs to specify a local measure $W_0(\phi)$, for instance of the Landau-Ginzburg type ($W_0(\phi)=e^{-A\phi^2-B\phi^4}$) or of the Ising type. Under a block spin transformation, the local measure changes according to:

$$W_{n+1}(\phi) \propto e^{\frac{\beta}{2}(\frac{c}{4})^{n+1}\phi^2} \times \int d\phi' W_n(\frac{(\phi-\phi')}{2})W_n(\frac{(\phi+\phi')}{2}) \ , \qquad (1)$$

This recursion formula can be reexpressed in

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Fourier representation.

$$R_{n+1}(k) \propto exp(-\frac{1}{2}\beta\frac{\partial^2}{\partial k^2})(R_n(\frac{\sqrt{c}k}{2}))^2 \qquad (2)$$

where $R_n(k)$ is the Fourier transform of $W_n(\phi)$ with an appropriate rescaling [5].

2. The Calculational Method

Recently, it was found [4] that finite dimensional approximations of degree l_{max}: $R_n(k) = 1 + a_{n,1}k^2 + a_{n,2}k^4 + ... + a_{n,l_{max}}k^{2l_{max}}$ provide very accurate results in the symmetric phase and moderately accurate results in the broken phase. In order to fix the ideas, one can calculate the critical exponent γ with three decimal points using $l_{max} = 15$ and with 13 decimal points with $l_{max} = 50$. The recursion formula for the $a_{n,m}$ reads $a_{n+1,m} = \frac{u_{n+1,m}}{u_{n+1,0}}$ with

$$\begin{aligned} u_{n+1,m} &= \sum_{l=m}^{l_{max}} (\sum_{p+q=l} a_{n,p}a_{n,q})\Gamma_{l,m} \\ \Gamma_{l,m} &= \frac{(2l)!}{(l-m)!(2m)!}(\frac{c}{4})^l(-\frac{1}{2}\beta)^{l-m} \ . \end{aligned} \qquad (3)$$

The initial condition for the Ising measure is $R_0 = cos(k)$. For the Landau-Ginsburg measure, the coefficients in the $k-$expansion need to be evaluated numerically. This is the only integral which needs to be calculated, after we only have *algebraic* manipulations. The effects of finite dimensional truncations decay faster than exponentially. If $\chi^{(l)}$ denotes the susceptibility calculated with $l_{max} = l$, then [4]

$$|\frac{\chi^{(l+1)} - \chi^{(l)}}{\chi^l}| \simeq l^{(-|s|l+q)} \qquad (4)$$

The volume effects can be controlled arbitrarily well in the symmetric phase where all the calculations which follow have been performed. The number $n(\beta, \Delta)$ of iterations necessary to calculate the susceptibility at fixed β, with a relative precision $|\frac{\chi_{n+1}-\chi_n}{\chi_n}| = \Delta$ can easily be estimated as

$$n(\beta, \Delta) = -(\frac{D}{2})(Log_2(\Delta) + \gamma Log_2(\beta_c - \beta)) \ . \qquad (5)$$

The main source of error comes from the round-off errors. Near criticality, $R_n(k)$ spends many iterations near the fixed point and the errors are amplified in the unstable direction. A simple calculation [4] shows that if δ is a typical round-off error (e.g. 10^{-16} in double precision), then the relative error on the susceptibility is of the order $(\beta_c - \beta)^{-1}\delta$. Numerical experiments confirm this estimate, however the detailed statistical properties has some intriguing features (non-gaussian distributions) which remain to be understood.

3. Numerical Results

The method described above can be combined with conventional expansions. In particular it allows us to calculate several hundred coefficients of the high-temperature expansion. In $D = 3$, it was found [6] that if we use a parametrization of the form

$$\chi = (\beta_c - \beta)^{-\gamma}(A_0 + A_1(\beta_c - \beta)^{\Delta} + ...) \qquad (6)$$

the A_0, A_1 are in general log-periodic functions of the form $\sum_l a_l(\beta_c - \beta)^{\frac{i2\pi l}{log\lambda}}$ where λ is the largest eigenvalue and $a_1 \neq 0$. In $D = 4$, the logarithmic corrections to the mean-field scaling laws can also be obtained from the high-temperature expansion. Minimizing the errors on $t(m) = ((r_m\beta_c - 1)m)^{-1}$, where r_m is the ratio of two successive HT coefficients, for $300 \leq m \leq 400$, we found [7] that

$$\chi \simeq (\beta_c - \beta)^{-\gamma}(A_0(|ln(\beta_c - \beta)|)^p + A_1) \qquad (7)$$

with $\gamma = 0.9997$ and $p = 0.3351$. This result is in good agreement with the classic field-theoretical result $\gamma = 1$ and $p = 1/3$.

Eq.(3) can be used to calculate numerically the renormalized coupling constants. Using the notation M_n for the total field $\sum\phi_x$ inside blocks of side 2^n and $< ... >_n$ for the average calculated inside these block, we define the dimensionless renormalized coupling

$$\lambda_{4,n} = \frac{- < M_n^4 >_n + 3(< M_n^2 >_n)^2}{2^n(\frac{< M_n^2 >_n}{2^n})^{\frac{D}{2}+2}} \qquad (8)$$

The numerator scales like 2^n while its two terms scale like 2^{2n} and as n increases, more and more significant digits get lost in the subtraction procedure. It is nevertheless possible to stabilize [4]

several digits of λ_4. We found that for $D = 3$, λ_4 reaches a finite non-zero limit at criticality $\lambda_4^* = 1.92786$ for a Ising measure. This property is sometimes refered to as hyperscaling. When one approaches criticality (which can easily be reformulated as "when the cut-off becomes large"), this limiting value is approached according to the approximate law

$$\lambda_4 - \lambda_4^* \simeq 1.68 \times (\beta_c - \beta)^{+0.43} \quad (9)$$

In $D = 4$, one can check the "triviality" of the theory. For an Ising measure, λ_4 reaches zero when β tends to β_c according to the approximate law [4]

$$(1/\lambda_4) \simeq -1.96 - 0.746 \times ln(\beta_c - \beta) \quad (10)$$

One can also use Eq.(3) to calculate approximate fixed points of the RG transformation. For this purpose, we start with an arbitrary measure and we fine-tune β until $R_n(k)$ stabilizes for a large number of iterations. This can be monitored in terms of ratio of successive coefficients. The approximate fixed point so obtained are fixed points for a particular value of β, however, it is possible to reabsorb this dependence by a rescaling of k. We have followed this procedure for a large class of models [9], namely $W_0(\phi) \propto exp^{-(\frac{1}{2}m^2\phi^2 + g\phi^{2p})}$ with $m^2 = \pm 1$ (single or double-well potentials), $p = 2, 3$ or 4 (coupling constants of positive, zero and negative dimensions when the cut-off is restored) and $g = 10$ or 0.1 (moderately large and small couplings). All the approximate rescaled fixed points $R^\star(\sqrt{\beta_c}k)$ we found turned out to be very close to a a universal function

$$U(k) = 1. - 0.35871 * k^2 + 0.05354 * k^4 ... \quad (11)$$

This universal function coincide in very good approximation with the function which can be obtained from the fixed point constructed with great accuracy by Koch and Wittwer [8]. Namely, we found that $|\delta u_l| < \frac{5\times 10^{-5}}{l! 2^l}$ where u_l are the coefficients of U. The use of this fixed point allows a very accurate determination [10] of the critical exponents appearing in Eq. (6): $\gamma = 1.2991407301586$ and$\Delta = 0.4259468589881$. Direct fits of the susceptibility confirm 11 decimal points of γ and 5 of Δ.

Work in progress involves the improvement of the hierarchical approximation, development of accurate methods in the broken phase and tests of perturbation theory. We are also considering hierarchical Fermi-Bose systems in order to test if it is possible to construct fully non-perturbative model without fine-tuning. In other words can the addition of hierarchical fermions with a suitably chosen set of bare couplings to bosons drive the boson measure "naturally" toward the stable manifold? Another question being investigated is: given the fixed point of Dyson model, can we calculate the susceptibility away from criticality? Or in other words, does the result of Koch and Wittwer [8] solve the model ? The answer to this question seems to be yes in low l_{max} cases.

REFERENCES

1. F. Dyson, Comm. Math. Phys. **12**, 91 (1969) ; G. Baker, Phys. Rev. B**5**, 2622 (1972).
2. Y. Meurice and G. Ordaz J. Phys. A **29**, L635 (1996).
3. K. Wilson, Phys. Rev. B. **4** 3185 (1971) ;
4. J.J Godina, Y. Meurice, B. Oktay and S. Niermann, Phys. Rev. D **57**, 6326 (1998).
5. Y. Meurice and G. Ordaz, J. Stat. Phys. **82**, 343 (1996).
6. Y. Meurice, G. Ordaz and V. G. J. Rodgers, Phys. Rev. Lett. **75**, 4555 (1995); Y. Meurice, S. Niermann, and G. Ordaz, J. Stat. Phys. **87**, 363 (1997).
7. J.J Godina, Y. Meurice, and S. Niermann, Nucl. Phys. B **519** [FS], 737 (1998).
8. H. Koch and P. Wittwer, Math. Phys. Electr. Jour. (http://mpej.unige.ch/), **1**, Paper 6 (1995).
9. J.J Godina, Y. Meurice, B. Oktay Phys. Rev. D **57**, R6581 (1998).
10. J.J Godina, Y. Meurice, B. Oktay, U. of Iowa preprint 98-2502